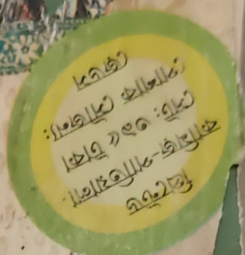
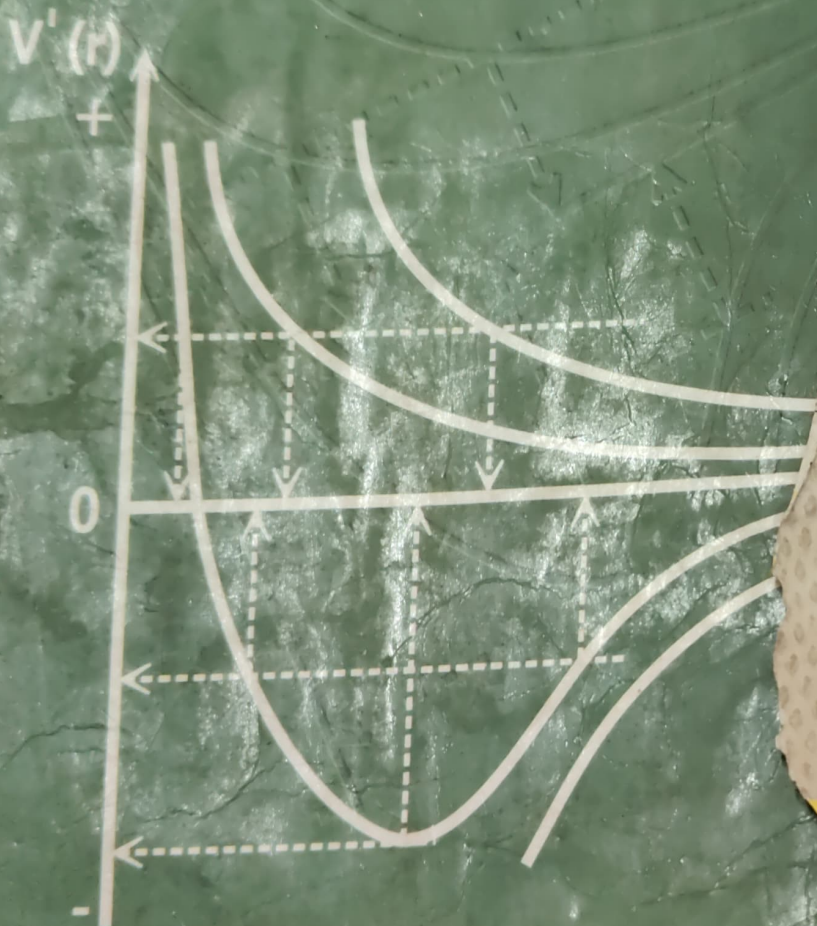


Pragati

CLASSICAL MECHANICS

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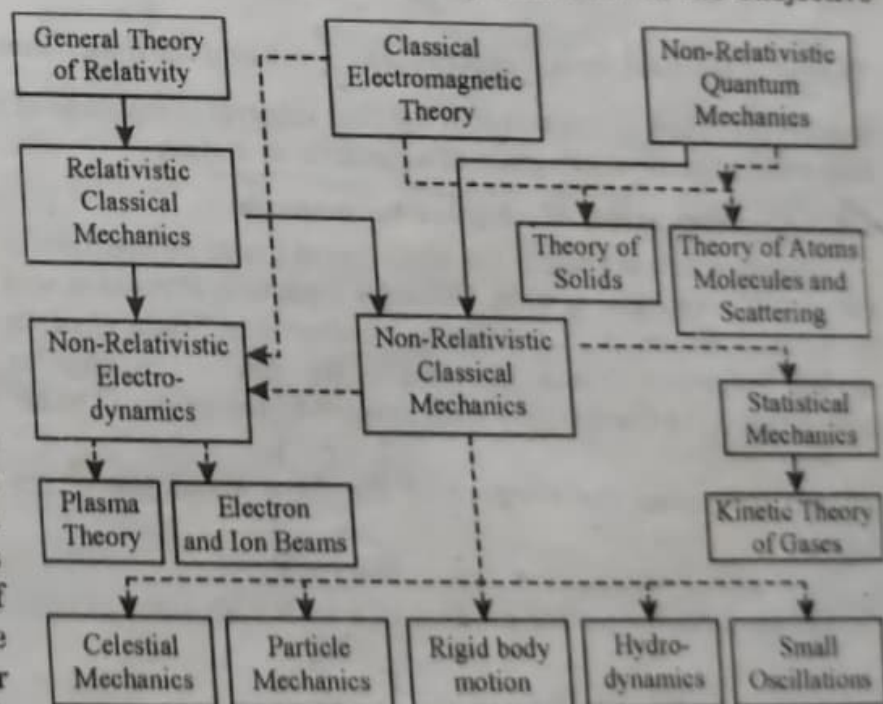
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Mechanics of a Particle and System of Particles

1.1. CLASSICAL MECHANICS AND OTHER THEORIES

Mechanics is the study of motion of physical bodies. The possible and actual motion of physical objects, whether large or small, fall under the domain of mechanics. The motion of celestial bodies (planets, stars etc.) path of an artillery shell, or of a space-satellite sent from earth to a planet are among its problems. So too is the analysis of the structure of an atom by way of motions of the particles constituting it. *Classical mechanics to-day has come into wide use to denote that part of mechanics where the objects in question are neither too big so that a close agreement between theory and experiment is desirable, nor too small interacting objects, more precisely, systems of an atomic scale.* These extreme cases are dealt within general theory of relativity and quantum mechanics respectively. Classical mechanics will also be interpreted to include the type of mechanics arising out of the special theory of relativity. We shall endeavour to outline the structure of classical mechanics and its applications in this book, starting essentially from the very fundamentals and primitive concepts and laws. Having thus established the pre-requisites, we shall switch over to more abstract approach of Lagrange and Hamilton which is known for its mathematical elegance and ingenuity. Towards the end of these chapters, we shall briefly comment on the subjective approach of this formulation.

We, now, digress to discuss the intimate relation of classical mechanics with other basic theories of physics. Recent theories, as will appear, are built on the solid foundation of classical mechanics, that is why its detailed exposition to every student opting for physics is so vitally important for a clear understanding of recent intricate theories. Figure illustrates this connection. A line (solid) connecting two boxes indicates that the branch of theoretical physics, to which the arrow points, is a particular case of the branch from which the line originates. The broken lines lead to various topics that we study in a particular theory.



1.2. CONSERVATION PRINCIPLES (LAWS)

The word 'conservation' applies in the sense of *constantness* when some characteristics of the motion of a system remains constant in time. In our physical world there exist a number of conservation principles or laws, some exact and some approximate. There are conservation laws

relating to energy, linear momentum, angular momentum, charge and various other quantities. They have been established as a result of extensive research on particle systems. Sometimes the laws seem to break down apparently, but subsequently it has appeared that we were wrong in framing the basic postulates. Conservation laws prove to be very powerful tools in solving mechanical problems in following respects :

- (i) Conservation laws are independent of the details of the trajectory and, often, of the details of a particular force. The laws are, therefore, a way of stating very general and significant consequence of equations of motion. Conservation laws assure us many times that some aspects of motion are impossible and must be left out.
- (ii) Conservation laws have an intimate relation with invariance. Their failure in certain cases may result in the discovery of new and not yet understood phenomena.
- (iii) Conservation laws tell us a great deal about the motion even if forces affecting motion are not known in advance.
- (iv) Even when the force is known exactly, a conservation law may provide a convenient method of solving the motion of the particle and particle systems. We exploit them in many ways to achieve this end.

1.3. MECHANICS OF A PARTICLE

We shall study the conservation laws for a particle in motion using Newtonian mechanics.

1.3-1. Conservation of linear momentum :

Conservation of linear momentum follows immediately from Newton's second law of motion in the form,

$$\frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt} = \mathbf{F} \quad \dots (1)$$

That is, if the total force $\mathbf{F}$ is zero then $\frac{d\mathbf{p}}{dt} = 0$ and the linear momentum is conserved. Obviously, the conservation law is the result of the first integral of equation of motion. Conservation laws are thus also called the first integrals of equations of motion.

1.3-2. Conservation of angular momentum :

Angular momentum is the analogue of linear momentum in the case of rotational motion. Consider a particle of mass m and linear momentum $\mathbf{p}$ at a position $\mathbf{r}$ relative to origin O of an inertial reference frame [fig. (1.1)]. We define the angular momentum $\mathbf{L}$ of the particle with respect to the origin O to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

and the torque as the moment of the force about the origin O , i.e.,

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}.$$

Let us form the vector product of $\mathbf{r}$ with both sides of equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt},$$

obtaining

$$\begin{aligned} \mathbf{N} = \mathbf{r} \times \mathbf{F} &= \mathbf{r} \times \frac{d\mathbf{p}}{dt} \\ &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) - \frac{d\mathbf{r}}{dt} \times \mathbf{p} \\ &= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) - \mathbf{v} \times m\mathbf{v}. \end{aligned}$$

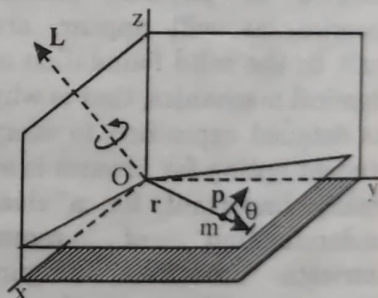


Fig. 1.1

The second term is zero, as both vectors are parallel and therefore

$$\begin{aligned} \mathbf{N} &= \frac{d}{dt} (\mathbf{r} \times \mathbf{p}) \\ &= \frac{d\mathbf{L}}{dt} \end{aligned} \quad \dots (2)$$

Thus time rate of change of the vector angular momentum of a particle is equal to the vector torque acting on it. Equation (2) is the analogue of Newton's second law of motion in the case of rotational motion. Conservation theorem of angular momentum follows at once from eq. (2) and can be stated as follows :

If the total torque $\mathbf{N} = 0$, then $\frac{d\mathbf{L}}{dt} = 0$ and therefore the angular momentum is conserved in the absence of an external torque.

Example : Show that the angular momentum is conserved in motion under central force. Let us consider a particle subjected to a central force of the form

$$\mathbf{F} = \hat{\mathbf{r}} F(r).$$

Torque of such a force is

$$\begin{aligned} \mathbf{N} &= \mathbf{r} \times \mathbf{F} \\ &= \mathbf{r} \times [\hat{\mathbf{r}} F(r)] = 0; \end{aligned}$$

so that for central forces $d\mathbf{L}/dt = 0$ and angular momentum is constant. This is why motion on account of central forces is always confined to a plane [fig. (1.2)].

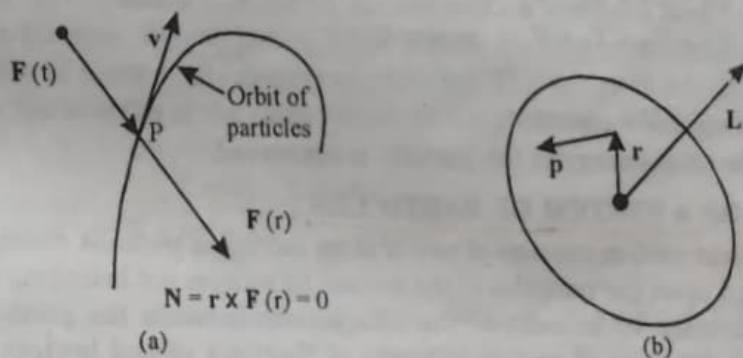


Fig. 1.2.

[Since $\mathbf{L}$ is constant in space in central force motion, the particle motion is confined to a plane.]

1.3-3. Conservation of Energy :

If the particle is acted upon by the forces which are conservative, that is, if the forces are derivable from scalar potential energy function in the manner

$$\mathbf{F} = -\nabla V,$$

then the total energy of the particle (kinetic + potential) is conserved.

Suppose, under the action of such a force, a particle moves from position 1 to position 2. Then work done by the particle will be,

$$\begin{aligned} W_{12} &= \int_1^2 \mathbf{F} \cdot d\mathbf{r} \\ &= \int_1^2 \frac{d\mathbf{p}}{dt} \cdot d\mathbf{r} \end{aligned}$$

If the particle moves $d\mathbf{r}$ distance in time dt with velocity $\dot{\mathbf{r}}$ then

and

$$\mathbf{p} = m\dot{\mathbf{r}}$$

$$d\mathbf{r} = \dot{\mathbf{r}} dt$$

so that

$$W_{12} = \int_1^2 \frac{d}{dt} (m\dot{\mathbf{r}}) \cdot \dot{\mathbf{r}} dt$$

$$= \int_1^2 \frac{d}{dt} \left(\frac{1}{2} m \dot{\mathbf{r}}^2 \right) dt$$

$$= T_2 - T_1,$$

where T_2, T_1 denote the kinetic energy of the particle at positions 2 and 1 respectively. ... (3)

Further, since

$$\mathbf{F} = -\nabla V,$$

we can write

$$W_{12} = \int_1^2 -\nabla V \cdot d\mathbf{r}$$

$$= \int_1^2 -\frac{dV}{dr} dr = -\int_1^2 dV$$

$$= V_1 - V_2.$$

From eqs. (3) and (4), we get

$$T_2 - T_1 = V_1 - V_2$$

or

$$T_2 + V_2 = T_1 + V_1 = \text{constant},$$

or in general

$$T + V = \text{constant},$$

which shows that the total energy of the particle is conserved.

1.4. MECHANICS OF A SYSTEM OF PARTICLES

When a mechanical system consists of two or more particles, we must distinguish between the *external forces* exerted upon the particles of the system by sources not belonging to the system, and the *internal forces* arising on account of the interactions between the particles of the system themselves. Thus (the equation of motion in terms of Newton's second law) can be easily written down (for a general system of N particles ;

$$m_i \mathbf{a}_i = \dot{\mathbf{p}}_i = \mathbf{F}_i^{(e)} + \sum_{j \neq i} \mathbf{F}_i^j, \quad i = 1, 2, \dots, N. \quad \dots (1)$$

From eq. (1), we obtain equation of motion of each particle, i , corresponding to each value of i ; they are N equations in all. Here $\mathbf{F}_i^{(e)}$ stands for the external force acting on i^{th} particle and $\mathbf{F}_i^j$ is the *internal force on the i^{th} particle due to j^{th} particle*. All the particles of the system exert forces on one another; hence the internal force on i^{th} particle must be the sum of forces due to all other particles = $\sum_{j=1}^N \mathbf{F}_i^j$ excluding the term $j = i$, since by definition $\mathbf{F}_i^i$ is obviously zero.

We shall modify eq. (1) by assuming that Newton's third law is valid for internal forces; that is, the force $\mathbf{F}_i^j$ must be equal and opposite in direction to the force $\mathbf{F}_j^i$ that the i^{th} particle exerts on the j^{th} particle. Vectorially

$$\mathbf{F}_i^j = -\mathbf{F}_j^i.$$

... (2)

It automatically implies that internal forces occur in pairs and act along line joining the two particles. Any combination of mutual forces must be zero then. This is referred to as the 'strong' form of Newton's third law. It is, however, not applicable to all physical systems, e.g., in systems involving moving charge, this law is violated. What is the implication of eq. (2) for a closed system? We know that if all possible actions and reactions are found totally confined within a system then such a system, by definition, is called a *closed system*. By third law, such a system must conserve the total linear momentum. Again, a closed system is not acted upon by external forces. Hence, by the first Newton's law, a closed system as a whole must act as an inertial frame.

Summing now over all particles of the system, we obtain the equation of motion of the system as a whole :

$$\begin{aligned}\sum_i \dot{p}_i &= \frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i \\ &= \sum_i \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{F}_i^j \\ &= \sum_i \mathbf{F}_i^{(e)},\end{aligned}\quad \dots (3)$$

since

$$\begin{aligned}\sum_{i,j} \mathbf{F}_i^j &= \sum_{i,j} \mathbf{F}_j^i, \\ &= \frac{1}{2} \sum_{i,j} [\mathbf{F}_i^j + \mathbf{F}_j^i] = 0\end{aligned}\quad \checkmark$$

as a consequence of eq. (2), where a prime on the summation symbol Σ means that the term $j = i$ is to be excluded from the sum. We can put eq. (3) in a more easily apprehensible form by introducing the concept of centre of mass of the system of particles. The vector $\mathbf{R}$ which locates the centre of mass is defined as the average of the radii vectors of the particles, weighted in proportion to their mass, i.e.,

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M},$$

where M is the total mass of the system $M = \sum_i m_i$. With the introduction of $\mathbf{R}$, eq. (3) assumes simple form. That is

$$\begin{aligned}M \frac{d^2 \mathbf{R}}{dt^2} &= \sum_i \mathbf{F}_i^{(e)} \\ &= \mathbf{F}^{(e)}\end{aligned}\quad \dots (4)$$

and the total linear momentum of the system is expressed as

$$\begin{aligned}\mathbf{P} &= \sum_i m_i \dot{\mathbf{r}}_i \\ &= \frac{d}{dt} \sum_i m_i \mathbf{r}_i \\ &= M \dot{\mathbf{R}}\end{aligned}\quad \dots (5)$$

1.4-1. Conservation of linear momentum :

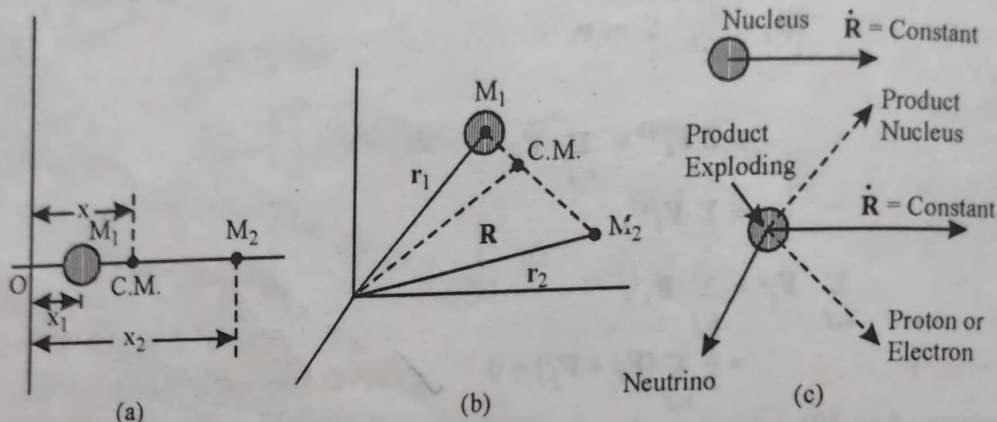
From eq. (5) rate of change of total linear momentum becomes

$$\dot{\mathbf{P}} = M \ddot{\mathbf{R}} = \mathbf{F}^{(e)},\quad \dots (6)$$

which defines two important characteristics of motion :

- (1) Centre of mass moves as if the total external force $\mathbf{F}^{(e)}$ acting on the entire mass of the system were concentrated at the centre of mass.
- (2) If the total external force vanishes, the total linear momentum is conserved.

Property (2) is the theorem of conservation of linear momentum for a system of particles. It also follows from this that since $\mathbf{P} = 0$, $\mathbf{P} = \text{constant}$ or $\dot{\mathbf{R}} = \text{constant}$, i.e., the centre of mass moves with constant velocity in the absence of external forces. An important example is that of uniform motion of a radioactive nucleus undergoing disintegration, ejecting different particles which move off in different directions in such a way that their centre of mass continues to move with constant velocity even after the disintegration [fig. (1.3c)].



$M_1 > M_2$
Two masses have a centre of mass in between at

$$X = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2}$$

For two masses M_1 and M_2 placed arbitrarily at $\mathbf{r}_1$ and $\mathbf{r}_2$, centre of mass is located at

$$X = \frac{M_1 \mathbf{r}_1 + M_2 \mathbf{r}_2}{M_1 + M_2}$$

Fig. 1.3.

1.4-2. Conservation theorem for angular momentum :

We obtain the total angular momentum of the system of particles by forming the cross product $(\mathbf{r}_i \times \mathbf{p}_i)$ for the i^{th} particle and summing over all particles. Let us multiply eq. (1) by $\mathbf{r}_i \times$ and sum over i :

$$\begin{aligned} \dot{\mathbf{L}} &= \frac{d}{dt} \sum_i (\mathbf{r}_i \times \mathbf{p}_i) = \sum_i [\mathbf{r}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{r}}_i \times \mathbf{p}_i] \\ &= \sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) \quad (\text{because } \dot{\mathbf{r}}_i \times \mathbf{p}_i = 0) \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_i^j. \end{aligned} \quad \dots (7)$$

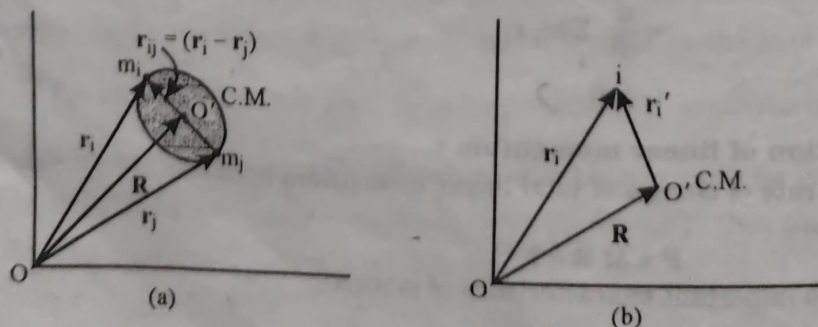


Fig. 1.4.

Second term denotes the sum of internal torques which, as we can prove*, vanishes if the interacting forces are Newtonian in character [eq. (2)]. We then have the important result

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \text{sum of external torques.} \quad \dots (8)$$

The time derivative of the total angular momentum is equal to the moment of the external forces about the given point. From this we have the conservation of angular momentum of a system of particles, viz.

If total external torque $\mathbf{N}^{(e)} = 0$, $\mathbf{L}$ is constant in time or conserved. ✓

Problem 1 : Express angular momentum of the system as the sum of angular momentum of motion of the centre of mass and angular momentum of the motion about the centre of mass.

We can decompose angular momentum of a system of particles, with reference to a given origin O , into two distinct parts :

(i) angular momentum of the system about the origin as if the total mass were concentrated at the centre of mass.

(ii) plus the angular momentum of the motion about the centre of mass.

It is done by observing that

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} \quad \text{and} \quad \mathbf{v}_i = \mathbf{v}_i' + \mathbf{v} \quad \dots (9)$$

in which $\mathbf{r}_i'$ and $\mathbf{v}_i'$ denote the radius vector and velocity of the i^{th} particle referred to centre of mass O' as the new origin and $\mathbf{V} = \dot{\mathbf{R}}$ is the velocity of the centre of mass relative to O (fig. 1.4b). Then

$$\begin{aligned} \mathbf{L} &= \sum_i m_i (\mathbf{r}_i' + \mathbf{R}) \times (\mathbf{v}_i' + \mathbf{v}) \\ &= \sum_i (\mathbf{R} \times m_i \mathbf{v}) + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' + (\sum_i m_i \mathbf{r}_i') \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} (\sum_i m_i \mathbf{r}_i') \end{aligned}$$

The last two terms vanish for both contain the factor $\sum_i m_i \mathbf{r}_i'$ which is equal to

$$\sum_i m_i \mathbf{r}_i' = \sum_i m_i (\mathbf{r}_i - \mathbf{R}) = \mathbf{MR} - \mathbf{MR} = 0,$$

from the definition of centre of mass. Also

$$\sum_i (\mathbf{R} \times m_i \mathbf{v}) = \mathbf{R} \times \mathbf{Mv}$$

so that

$$\mathbf{L} = \mathbf{R} \times \mathbf{Mv} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i' \quad \dots (10)$$

* Let the sum of all internal torques be represented by $\mathbf{N}_{int}$, then

$$\mathbf{N}_{int} = \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_i^j = \sum_{i,j} \mathbf{r}_j \times \mathbf{F}_j^i$$

in which we have merely relabelled dummy indices i and j (dummy index is one which occurs twice in a term and stands for summation). This does not make any difference since in both the summation all the particles and associated forces have been included. Hence

$$\mathbf{N}_{int} = \frac{1}{2} \sum_{i,j} [\mathbf{r}_i \times \mathbf{F}_i^j + \mathbf{r}_j \times \mathbf{F}_j^i].$$

If we assume that the forces are Newtonian i.e., $\mathbf{F}_i^j = -\mathbf{F}_j^i$, this equation modifies to

$$\mathbf{N}_{int} = \frac{1}{2} \sum_{i,j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_i^j.$$

But $(\mathbf{r}_i - \mathbf{r}_j)$ is identical with the vector $\mathbf{r}_{ij}$ from the j^{th} to i^{th} particle which is parallel to the line of action of $\mathbf{F}_i^j$ [fig. (1.4)]; hence

$$\mathbf{r}_{ij} \times \mathbf{F}_i^j = 0; \text{ therefore } \mathbf{N}_{int} = 0,$$

and the net internal torque vanishes. ✓

which is the desired expression. If centre of mass is at rest, or moves parallel to $\mathbf{R}$, first term is zero and then $\mathbf{L}$ will be independent of the reference point and consists entirely the angular momentum of the system about the centre of mass :

$$\mathbf{L}' = \sum_i \mathbf{r}_i' \times \mathbf{p}_i', \quad \dots (11)$$

Importance of eq. (10) lies in the fact that we can treat the motion of the system in terms of motion of centre of mass in a certain reference frame and motion about the centre of mass independently of each other provided the total torque acting on the system be separated in a like manner. That this is possible can be seen by writing the total torque $\mathbf{N}^{(e)}$ with the aid of eq. (9) :

$$\begin{aligned} \mathbf{N}^{(e)} &= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \sum_i (\mathbf{r}_i' + \mathbf{R}) \times \mathbf{F}_i^{(e)} \\ &= \sum_i \mathbf{r}_i' \times \mathbf{F}_i^{(e)} + \mathbf{R} \times \sum_i \mathbf{F}_i^{(e)} \\ &= \mathbf{N}'^{(e)} + \mathbf{N}_0^{(e)}, \end{aligned} \quad \dots (12)$$

where $\mathbf{N}'^{(e)}$ is the total torque about the centre of mass as origin and $\mathbf{N}_0^{(e)}$, that about the origin O . Thus we obtain two independent equations of motion— one for describing the motion of centre of mass in a reference co-ordinate system and other in the centre of mass-co-ordinate system. We write

$$\begin{aligned} \mathbf{N}_0^{(e)} &= \mathbf{R} \times \mathbf{F}^{(e)} \\ &= \frac{d}{dt} (\mathbf{R} \times \mathbf{M}\mathbf{v}) \end{aligned} \quad \dots (13)$$

and

$$\begin{aligned} \mathbf{N}'^{(e)} &= \frac{d}{dt} (\mathbf{r}_i' \times \mathbf{p}_i') \\ &= \frac{d\mathbf{L}'}{dt} \end{aligned} \quad \dots (14)$$

Problem 2 : Show that kinetic energy can be expressed as the sum of the kinetic energy of motion of the centre of mass and the kinetic energy of motion about the centre of mass.

We write, in the laboratory coordinate system, the expression for kinetic energy is

$$K.E. = \frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} \sum_i m_i \mathbf{v}_i \cdot \mathbf{v}_i$$

From equation (9), putting for $\mathbf{v}_i$ in terms of the velocity of centre of mass $\mathbf{v}$ and about the centre of mass $\mathbf{v}_i'$, we get

$$\begin{aligned} K.E. &= \frac{1}{2} \sum_i m_i (\mathbf{v} + \mathbf{v}_i') \cdot (\mathbf{v} + \mathbf{v}_i') \\ &= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 + \sum_i m_i \mathbf{v}_i' \cdot \mathbf{v} \end{aligned}$$

We can show that the term

$$\sum_i m_i \mathbf{v}_i' \cdot \mathbf{v} = \sum_i m_i \frac{d\mathbf{r}_i'}{dt} \cdot \mathbf{v} = \frac{d}{dt} (\sum_i m_i \mathbf{r}_i') \cdot \mathbf{v} = 0$$

so that

$$\begin{aligned} K.E. &= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i v_i'^2 \\ &= K.E. \text{ of motion of centre of mass} + K.E. \text{ of motion about} \\ &\quad \text{the centre of mass} \end{aligned}$$

1.4-3. Conservation of Energy :

Conservation of the mechanical energy of a system is to be expected when the forces acting on the particles are conservative ones. Such forces were found to be derivable from a potential energy function, which in the case of a system of particles can be decomposed into two parts,

$$V = V^{(e)} + V^{int}, \quad \dots (15)$$

such that

$$\mathbf{F}_i^{(e)} = -\nabla_i V^{(e)} \quad \text{and} \quad \mathbf{F}_i^{int} = -\nabla_i V^{int}, \quad \dots (16)$$

where the subscript i on the del operator indicates that differentiation implied by ∇ -operation is to be carried with respect to the co-ordinates of i th particle and corresponding potential energy function is to be taken at the position of i th particle. Potential energy function for external applied forces is necessarily the function of the co-ordinates of the position of the particle. What about the internal potential energy function V^{int} ? If the interacting forces, to which it gives rise, are Newtonian, then V^{int} must be a function of the relative distance between the two interacting particles i.e.,

$$\begin{aligned} V^{int} &= \sum_j V_i^j = \sum_j V_j^i (|\mathbf{r}_i - \mathbf{r}_j|) \\ &= \sum_j V_i^j (|\mathbf{r}_j - \mathbf{r}_i|) \\ &= \sum_j V_j^i (|\mathbf{r}_i - \mathbf{r}_j|) = V_j^{int}. \end{aligned}$$

Since potential energies are additive algebraically, we obtain total (mutual) potential energy as

$$\begin{aligned} V^{int} &= \sum_i V_i^{int} \\ &= \frac{1}{2} \sum_{i,j} V_j^i (|\mathbf{r}_i - \mathbf{r}_j|); \end{aligned}$$

a factor $\frac{1}{2}$ comes because of the fact that while summing the mutual potential energies, a pair of particles i, j , appears twice : once for i^{th} due to j^{th} and secondly for j^{th} due to i^{th} particle, whereas both are the same and should appear only once.

Proof : Now we proceed to prove conservation of total energy for a system of particles. Suppose i^{th} particle is displaced through $d\mathbf{r}_i$, then the amount of work done by the total force $\mathbf{F}_i$ acting on the particle is $\mathbf{F}_i \cdot d\mathbf{r}_i$ which can also be expressed as

$$\begin{aligned} m_i \ddot{\mathbf{r}}_i \cdot d\mathbf{r}_i &= \mathbf{F}_i \cdot d\mathbf{r}_i \\ &= (\mathbf{F}_i^e + \sum_j \mathbf{F}_i^j) \cdot d\mathbf{r}_i \\ &= \mathbf{F}_i^e \cdot d\mathbf{r}_i + \sum_j \mathbf{F}_i^j \cdot d\mathbf{r}_i \end{aligned} \quad \dots (17)$$

Since
eq. (17) can be written as

$$m_i \ddot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i dt = \mathbf{F}_i^e \cdot d\mathbf{r}_i + \sum_j \mathbf{F}_i^j \cdot d\mathbf{r}_i$$

$$\text{or} \quad \frac{d}{dt} \left(\frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) dt = \mathbf{F}_i^e \cdot d\mathbf{r}_i + \sum_j \mathbf{F}_i^j \cdot d\mathbf{r}_i$$

Writing the above equation for the system of particles, we get

$$\frac{d}{dt} \left(\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) dt = \sum_i \mathbf{F}_i^e \cdot d\mathbf{r}_i + \sum_{i,j} \mathbf{F}_i^j \cdot d\mathbf{r}_i \quad \dots (18)$$

We shall first take up the right hand side of eq. (18). The second term is

$$\sum_{i,j} \mathbf{F}_i^j \cdot d\mathbf{r}_i = \frac{1}{2} \sum_{i,j} (\mathbf{F}_i^j \cdot d\mathbf{r}_i + \mathbf{F}_j^i \cdot d\mathbf{r}_j) \quad \dots (19)$$

If we assume that the forces obey Newton's third law, (e.g., central forces) then

$$\mathbf{F}_i^j = -\mathbf{F}_j^i.$$

Also these forces are derivable from scalar potential function V_i^j or in other words they are conservative. We have shown above that V_i^j depends only on the interparticle distance i.e.

$$V_i^j = V_i^j(|\mathbf{r}_i - \mathbf{r}_j|)$$

and as it is scalar, we have

$$V_i^j = V_j^i.$$

Therefore,

$$\mathbf{F}_i^j = -\nabla_i V_i^j$$

$$\mathbf{F}_j^i = -\nabla_j V_i^j$$

Denoting,

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j,$$

and gradient with respect to $\mathbf{r}_{ij}$ as ∇_{ij} , we arrive at

$$\begin{aligned} \mathbf{F}_i^j &= -\nabla_i V_i^j = -\frac{\partial}{\partial \mathbf{r}_i} (V_i^j) = -\frac{\partial}{\partial \mathbf{r}_{ij}} V_i^j \left(\frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{r}_i} \right) \\ &= -\nabla_{ij} V_i^j \cdot \left(\frac{\partial}{\partial \mathbf{r}_i} (\mathbf{r}_i - \mathbf{r}_j) \right) \\ &= -\nabla_{ij} V_i^j. \end{aligned} \quad \dots (20)$$

Also

$$\begin{aligned} \mathbf{F}_j^i &= -\nabla_j V_i^j = -\frac{\partial}{\partial \mathbf{r}_j} (V_i^j) = -\frac{\partial V_i^j}{\partial \mathbf{r}_{ij}} \left(\frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{r}_j} \right) \\ &= -\nabla_{ij} V_i^j \left(\frac{\partial}{\partial \mathbf{r}_j} (\mathbf{r}_i - \mathbf{r}_j) \right) \\ &= \nabla_{ij} V_i^j. \end{aligned} \quad \dots (21)$$

Putting eqs. (20) and (21) into eq. (19), we get

$$\begin{aligned} \sum_{i,j} \mathbf{F}_i^j \cdot d\mathbf{r}_i &= \frac{1}{2} \sum_{i,j} (-\nabla_{ij} V_i^j \cdot d\mathbf{r}_i + \nabla_{ij} V_i^j \cdot d\mathbf{r}_j) \\ &= -\frac{1}{2} \sum_{i,j} \nabla_{ij} V_i^j \cdot d(\mathbf{r}_i - \mathbf{r}_j) \\ &= -\frac{1}{2} \sum_{i,j} \nabla_{ij} V_i^j \cdot d\mathbf{r}_{ij}. \end{aligned} \quad \dots (22)$$

If the external force is also conservative, then $\mathbf{F}_i^e$ can also be expressed as gradient of scalar V_i^e , that is

$$\mathbf{F}_i^e = -\nabla_i V_i^e. \quad \dots (23)$$

With eqs. (22) and (23), we can write eq. (18) as

$$\frac{d}{dt} \left(\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) dt = -\sum_i \nabla_i V_i^e \cdot d\mathbf{r}_i - \frac{1}{2} \sum_{i,j} \nabla_{ij} V_i^j \cdot d\mathbf{r}_{ij}$$

which on integration gives

$$\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = - \sum_i V_i^e - \frac{1}{2} \sum_{i,j} V_{i,j}^j + \text{constant}$$

or

$$\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 + \sum_i V_i^e + \frac{1}{2} \sum_{i,j} V_{i,j}^j = \text{constant}$$

or

$$T + V = \text{constant} \quad \dots (24)$$

where

$$V = \sum_i V_i^e + \frac{1}{2} \sum_{i,j} V_{i,j}^j$$

= external potential energy of the system + internal potential energy of the system

Eq. (24) shows that if the external forces are derivable from a scalar potential function and if the internal forces are central, then total energy (kinetic + potential) of the system is conserved.

1.5. CONSTRAINED MOTION, CONSTRAINTS AND DEGREES OF FREEDOM

Constraints : *अवरोध*

A *constrained motion* is a motion which cannot proceed arbitrarily in any manner. Particle motion is restricted to occur only, for example, along some specified path, or on a surface (plane or curved) arbitrarily oriented in space. Motion along a specified path is the simplest example of a constrained motion. Here, one co-ordinate is sufficient to describe the motion in contrast to the situation where the particle is free to move in space and then three co-ordinates are needed to describe its motion. Thus imposing constraints on a mechanical system is to simplify the mathematical description. When a particle (bead) is made to slide on a wire, constraints require that the position of the bead lie on the wire. Condition imposed on the system by the constraints can, in most cases, be written down mathematically as a relation satisfied by the co-ordinates of the particle at any time. This is the way in which constraints reduce the number of co-ordinates needed to specify the configuration of a system.

☆ **Example 1 :** Let us consider the motion of a simple pendulum confined to move in the vertical plane. We would need only two co-ordinates (cartesian co-ordinates x and y or polar co-ordinates r and θ with respect to the point of suspension, O , as origin) to locate the position of the bob in motion. However, motion of the bob is not free but takes place under a constraint that the distance l of the bob is to remain the same from O at all time. This condition imposed by the constraint can be expressed in the form of an equation either between x and y or r and θ :

Constraints

$$\begin{aligned} x^2 + y^2 &= l^2, \\ r &= l. \end{aligned} \quad \dots (1)$$

or

In plane polar co-ordinates, the equation looks simpler. Again one co-ordinate θ , in polar co-ordinates, would suffice to describe the motion. Note that we have utilised eq. (1) to reduce the number of co-ordinates which otherwise would have been two.

☆ **Example 2 :** Let us take up another example. A particle moving in space requires three co-ordinates to determine its position at any instant. If we restrict its movement on the surface of a sphere, there exists a relation between these co-ordinates. Again we shall see that spherical polar co-ordinates can be used with advantage :

$$\begin{aligned} x^2 + y^2 + z^2 &= a^2, \\ r &= a \end{aligned} \quad \dots (2)$$

or

in which a is the radius of the sphere. Each of eqs. (2) is the equation of the surface of the sphere with its centre at origin. We may use the equation of constraints to eliminate one co-ordinate from the set of three co-ordinates and we are left with two co-ordinates θ and ϕ to describe the position of the particle completely. This can be performed conveniently by using spherical polar co-ordinates when we select θ and ϕ as independent co-ordinates.

☆ **Example 3 :** We take up a third example of double pendulum shown in fig. (1.5) moving in a vertical plane (which is a two-particle system connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles). We would require four co-ordinates (two for each particle) to describe the system completely : but two of them are eliminated by the equations of constraints *viz.* distances to particle 1 should be constant when measured from point of suspension O and from particle 2. Then two convenient co-ordinates would be the angles θ_1 and θ_2 shown in the figure.

☆ **Example 4 :** As a last and final example, we consider a rigid body. A rigid body is defined as a system of particles in which the relative distance of the constituent particles are fixed and cannot vary with time. In this case, constraints are expressed by equations of the form

$$r_{ij} = c_{ij},$$

in which c_{ij} are constants and r_{ij} denote the distance between i^{th} and j^{th} particles. In terms of co-ordinates $\mathbf{r}_i (x_i, y_i, z_i)$ and $\mathbf{r}_j (x_j, y_j, z_j)$ with respect to the origin we have these conditions expressed as :

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = (c_{ij})^2. \quad \dots (3)$$

All these instances serve to demonstrate that each constraint, which can be expressed in the form of an equation like (1), (2), (3) or in the general case of an N particle system as an equation connecting the co-ordinates of the particles having the form

$$f(x_1, y_1, z_1; x_2, y_2, z_2; \dots; x_N, y_N, z_N; t) = 0 \quad \dots (4)$$

where time t may occur in case of constraints which may vary with time, and enables us to eliminate one of the co-ordinates by choosing co-ordinate in such a manner that it is held constant by the constraint. For a rigid body containing N particles, there are $\frac{1}{2} N(N-1)$ pairs of particles.

It is not difficult to show that it is sufficient to specify the mutual distances of the $(3N-6)$ pairs, if $N > 3$. Hence we can replace the $3N$ cartesian co-ordinates originally needed, had the system been free from constraints, by 3 co-ordinates of centre of mass and 3 co-ordinates describing the orientation of the body (refer to chapter on rigid body motion). Now $(3N-6)$ distances are constant and the problem is reduced to one of finding the motion in terms of only 3 plus 3 = 6 co-ordinates.

Holonomic and Non-holonomic constraints :

Constraints that can be expressed in the form of eq. (4) are called *holonomic*.† All the above examples involve holonomic constraints. Constraints not expressible in this fashion are termed *non-holonomic*. The motion of the particle placed on the surface of sphere under the action of gravitational force is bound by non-holonomic constraint, for it can be expressed as an inequality

$$r^2 - a^2 \geq 0.$$

Equality sign holds until the particle rolls on the sphere and when it leaves the sphere, we must have $r^2 - a^2 > 0$. The constraints involved in the motion of the molecules in a gas container are non-holonomic. An object rolling on a rough surface without slipping involves non-holonomic constraint in the description of its motion.

Sometimes the constraints are specified by a restriction on the velocities rather than on co-ordinates and still the constraints are holonomic. The fact is that the velocity equation is integrable to yield a relation of the type (4). For example, let us consider a cylinder of radius a , rolling and sliding down an inclined plane, with its axis always horizontal. The position of the cylinder can be located by two co-ordinates s and θ where s is the distance of the axis of cylinder measured along the inclined plane that the cylinder has moved and θ is the angle that a fixed radius in the cylinder has rotated from the radius to the point of contact with the plane. If the cylinder happens to roll without slipping, then the velocities $\dot{s}$ and $\dot{\theta}$ are such that

$$\dot{s} = a\dot{\theta}$$

† Derived from the word 'holos' which means integer in Greek and 'whole' or 'integrable' in Latin.

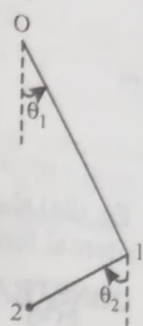


Fig. 1.5.

$$\frac{ds}{dt} = a \frac{d\theta}{dt},$$

or which on integration yields a relation of the type (4) :

$$s - a\theta = \text{constant.} \quad \dots (5)$$

There are, however, systems in which equations among velocities cannot be integrated to give a relation between co-ordinates as above. Such systems are then governed by *non-holonomic constraints*.

Scleronomic and Rheonomic Constraints :

If the constraints are independent of time, they are termed as *scleronomic* but if they contain time explicitly, they are called *rheonomic*. A bead sliding on a moving wire is an example of *rheonomic constraint*.

Therefore a constraint is

- | | | | | |
|----------|--------|----------------------|---|---|
| (i) | Either | <i>Scleronomic</i> | : | Constraint relations do not explicitly depend on time |
| | or | <i>Rheonomic</i> | : | Constraint relations depend explicitly on time |
| and (ii) | Either | <i>Holonomic</i> | : | Constraint relations are or can be made independent of velocities |
| | or | <i>Non-holonomic</i> | : | Constraint relations are not holonomic. This relations are irreducible functions of velocities. |

Further, if constraint relations are in the form of equations, they are termed as *bilateral*, but if relations are expressed in the form of inequalities then constraints are termed as *unilateral*. There is another classification based on whether constraint forces do work or not. If constraint forces do work and total mechanical energy is not conserved then constraints are called *dissipative*. But if constraint forces do not do any work and total mechanical energy of the system is conserved while performing the constrained motion, they are called *conservative*.

Constraints in Some cases :

System		Type of Constraint
1.	Rigid body	Conservative in nature, scleronomic, holonomic, bilateral
2.	Deformable bodies	Rheonomic, holonomic, bilateral, dissipative
3.	Simple pendulum with rigid support	Scleronomic, holonomic, bilateral, conservative
4.	Pendulum with variable length	Rheonomic, holonomic, bilateral, dissipative
5.	Rolling without sliding	non-holonomic
6.	Gas filled hollow sphere	Scleronomic, holonomic, conservative, unilateral
7.	An expanding or contracting spherical container of gas	Rheonomic, holonomic, unilateral, dissipative

FORCES OF CONSTRAINTS :

There is one significant point to note. Constraints not only interfere with the solution of the problems in that the co-ordinates are no longer independent but they are always associated with the forces by virtue of which they restrict the motion of the system. Such forces are termed the *forces of constraints*. We generally formulate the laws of mechanics in a way that the work done by the forces of constraints is zero when the system is in motion. By this we do not mean that the physical motion should be allowed to happen in a way not consistent with constraints; but we only require our formulation to sidetrack the effect of forces of constraints without violating them.]

Question arises how to choose a suitable set of generalised co-ordinates in a given situation ? In doing so we must be guided by the following three principles :

- (i) Their values determine the configuration of the system.
- (ii) They may be varied arbitrarily and independently of each other, without violating the constraints on the system.
- (iii) There is no uniqueness in the choice of generalised co-ordinates. Then our choice should fall on a set of co-ordinates that will give us a reasonable mathematical simplification of the problem.

Notation for generalised co-ordinates : Generalised co-ordinates are designated by letter q with numerical subscripts; $q_1, q_2, \dots, q_n$ represent a set of n generalised co-ordinates; or, alternatively, by a letter subscript to q and specifying within brackets the numerical values that the letter subscript is allowed to take, e.g., q_j ($j = 1, 2, \dots, n$). When we switch over to describe a specific problem, the symbols $q_1, q_2, \dots$ correspond to co-ordinates that we choose to describe the motion. Thus when a particle moves in a plane, it may be described by cartesian co-ordinates x, y or the polar co-ordinates, r, θ and so on, and we write :

$$\begin{aligned} q_1 &= x & \text{or} & & q_1 &= r = \sqrt{(x^2 + y^2)}, \\ q_2 &= y & & & q_2 &= \theta = \tan^{-1} \frac{y}{x}. \end{aligned} \quad \dots (8)$$

When the problem involves some spherical symmetry, it is suitable to use spherical co-ordinates :

$$\begin{aligned} q_1 &= r = (x^2 + y^2 + z^2)^{1/2}, \\ q_2 &= \theta = \cot^{-1} \frac{z}{(x^2 + y^2)^{1/2}} \\ q_3 &= \phi = \tan^{-1} \frac{y}{x}, \end{aligned} \quad \dots (9)$$

If it is preferred to accept a co-ordinate system moving uniformly with velocity v in x -direction, generalised co-ordinates are

$$\begin{aligned} q_1 &= x - \dot{x}t. \\ q_2 &= y & \dot{x} &= v = \text{constant}. \\ q_3 &= z. \end{aligned} \quad \dots (10)$$

For a rod lying on a plane surface, capable of taking any orientation, the suitable choice of co-ordinates to describe the configuration of the rod will be the cartesian co-ordinates ξ and η to locate any point A of the rod and angle θ indicating the orientation of the rod with respect to the co-ordinates axes OX and OY .

Then

$$\begin{aligned} q_1 &= \xi, \\ q_2 &= \eta, \\ q_3 &= \theta. \end{aligned} \quad \dots (11)$$

If x, y denote the cartesian co-ordinates of any other point B of the rod, distant r from the previous fixed point A , we have the connection between x, y and ξ, η, θ as

$$\begin{aligned} x &= \xi + r \cos \theta & y &= \eta + r \sin \theta \\ &= q_1 + r \cos q_3 & &= q_2 + r \sin q_3 \end{aligned} \quad \dots (12)$$

relative to axes OXY . In the transformation relations, r is simply a number. It cannot be treated as distinct co-ordinate; for, r cannot be varied without violating the constraint that the distance of two particles must remain constant in time. That is why we suppress explicit reference to such numbers.

In general, we can always express generalised co-ordinates as some functions of cartesian co-ordinates, and possibly function of time (*cf.* eqs. 8, 9, 10, 11, 12):

[illegible]

for a system of N particles free from constraints which require the specification of $3N$ generalised co-ordinates. Eqs. (13) are then the *transformation equations* from a set of $3N$ cartesian co-ordinates to $3N$ generalised co-ordinates. If constraints are present, the number of generalised co-ordinates will be reduced accordingly. And, since we accept the co-ordinates $q_1, q_2, \dots, q_{3N}$ as specifying the configuration of the system, it must be possible to express cartesian co-ordinates in terms of them and vice-versa :

[illegible]

The necessary and sufficient condition that the transformation from a set of co-ordinates q_j ($j = 1, 2, \dots, 3N$) to the set $(x_1, y_1, \dots, z_N)$ is effective is that the *Jacobian* determinant J of eqns. (13) be different from zero at all points, i.e.,

$$J \left(\frac{q_1, q_2, \dots, q_{3N}}{x_1, y_1, \dots, z_N} \right) \equiv \left(\frac{\partial (q_1, q_2, \dots, q_{3N})}{\partial (x_1, y_1, \dots, z_N)} \right) = \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \dots & \frac{\partial q_{3N}}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} & \dots & \frac{\partial q_{3N}}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial q_1}{\partial z_N} & \frac{\partial q_2}{\partial z_N} & \dots & \frac{\partial q_{3N}}{\partial z_N} \end{vmatrix} \neq 0$$

If this condition fails to hold, eqs. (13) do not define a consistent set of generalised co-ordinates. If the Jacobian determinant does not vanish, the co-ordinates q_j are as effective as the cartesian co-ordinates in describing the kinematical motion of the system and are most convenient to use. ~~Some examples of the mechanical systems and the generalised co-ordinates to be associated~~ are given below :

System	Generalised Co-ordinates
1. <u>Simple pendulum</u>	θ ; the angle which the pendulum makes with vertical line through point of suspension..
2. Fly-wheel	θ ; the angle between a definite radius of the fly wheel and fixed line perpendicular to the axis.
3. Particles on the surface of a sphere.	θ, ϕ ; the usual polar angle of a point on the sphere.
4. Beads of an abacus	x ; the cartesian co-ordinate along the horizontal wire.
5. Hydrogen molecule	x, y, z, ϕ, ψ ; x, y, z are the cartesian co-ordinates of the centre of molecule ; ϕ and ψ the angles of rotation about two mutually perpendicular axes through centre.
6. Particles moving on inside surface of a cone.	r, θ ; r the radius vector drawn from the vertex as origin to the position of the particle and angle θ of the radius vector with a fixed slant edge of the cone.

1.7. GENERALISED NOTATIONS

We shall now develop the usual notation for displacements, velocity, acceleration, momentum, force, potential energy in terms of generalised co-ordinates. We shall find them useful while deriving Lagrange's equations of motion.

(1) **Generalised Displacement** : Let us consider a small displacement of an N -particle system defined by changes $\delta \mathbf{r}_i$ in cartesian co-ordinates $\mathbf{r}_i$ ($i = 1, 2, \dots, N$) with time t held fixed. For the sake of simplicity we may consider an arbitrary virtual displacement $\delta \mathbf{r}_i$, then since $\mathbf{r}_i$ are functions of generalised co-ordinates defined by equation;

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_{3N}, t)$$

we have*

$$\delta \mathbf{r}_i = \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (\text{as } \delta t = 0) \quad \dots (15)$$

We have chosen $\mathbf{r}_i$ to represent the $3N$ co-ordinates ($x_1, y_1, z_1, \dots, z_N$) for notational convenience; each $\mathbf{r}_i$ is equivalent to three component co-ordinates x_1, y_1, z_1 and so on.

δq_j are called the generalised displacements or virtual arbitrary displacements. If q_j is an angle co-ordinate, δq_j is an angular displacement.

(2) **Generalised Velocity** : Velocity may be described in terms of time derivative $\dot{q}_j$ of the generalised co-ordinate q_j , which is then called generalised velocity associated with a particular co-ordinate q_j . We have for an unconstrained system :

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q_1, q_2, \dots, q_{3N}, t).$$

Then

$$\dot{\mathbf{r}}_i = \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \quad \dots (16)$$

If the N -system contains k constraints, the number of generalised co-ordinates is $3N - k = f$ and in that case

$$\dot{\mathbf{r}}_i = \sum_{j=1}^f \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \quad \dots (16')$$

Note that if the generalised co-ordinate q_j involves both cartesian and angle co-ordinates, the generalised velocity associated with a cartesian co-ordinate x is just the corresponding linear velocity $\dot{x}$ while generalised velocity with an angular co-ordinate θ is the corresponding angular velocity. If a generalised co-ordinate has the dimensions of momentum, the generalised velocity will have the dimensions of force and so on.

From eq. (16), the cartesian components of velocity $\dot{\mathbf{r}}_i$ are seen to be the linear functions of the generalised velocity components no matter how the generalised co-ordinates are defined. This means that it is easy to express velocities in generalised co-ordinates. The term $\partial \mathbf{r}_i / \partial t$ appears only when there are constraints depending on time or in some cases where it is convenient to introduce moving co-ordinate axes.

(3) **Generalised Acceleration** : Components of accelerations are given by differentiating eq. (16) or (16)', as the case may be.

Equation (16) is

$$\dot{\mathbf{r}}_i = \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}$$

*From Eulers' theorem

$$X = X(x_1, x_2, x_3) \\ dX = \frac{\partial X}{\partial x_1} dx_1 + \frac{\partial X}{\partial x_2} dx_2 + \frac{\partial X}{\partial x_3} dx_3 = \sum_{j=1}^3 \frac{\partial X}{\partial x_j} dx_j$$

Differentiating it again w.r.t. time, we get

$$\begin{aligned}\ddot{\mathbf{r}}_i &= \frac{d}{dt} \left(\sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right) \\ &= \sum_{j=1}^{3N} \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \ddot{q}_j + \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial t} \right) \\ &= \sum_{j=1}^{3N} \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \ddot{q}_j + \frac{\partial \dot{\mathbf{r}}_i}{\partial t}\end{aligned}$$

Putting for $\dot{\mathbf{r}}_i$ from eq. (16), on changing index j to k , we get

$$\begin{aligned}\ddot{\mathbf{r}}_i &= \sum_{j=1}^{3N} \frac{\partial}{\partial q_j} \left(\sum_{k=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right) \dot{q}_j \\ &\quad + \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \ddot{q}_j + \frac{\partial}{\partial t} \left(\sum_{k=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right) \\ &= \sum_{j=1}^{3N} \sum_{k=1}^{3N} \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k \dot{q}_j + \sum_{j=1}^{3N} \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t} \dot{q}_j + \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \ddot{q}_j + \sum_{k=1}^{3N} \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t^2} \\ \ddot{\mathbf{r}}_i &= \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \ddot{q}_j + \sum_{j=1}^{3N} \sum_{k=1}^{3N} \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_j \dot{q}_k + 2 \sum_{j=1}^{3N} \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t} \dot{q}_j + \frac{\partial^2 \mathbf{r}_i}{\partial t^2}, \quad \dots (17)\end{aligned}$$

The cartesian components of acceleration are not linear functions of components of generalised accelerations $\ddot{q}_j$ alone, but depend quadratically and linearly on the generalised velocity components $\dot{q}_j$ as well. However, in the new approach devised by Lagrange, the computation of second derivatives of generalised co-ordinates is not required.

(4) Generalised Momentum : Let us first write down an expression for kinetic energy in terms of generalised velocities. The kinetic energy T of a system of N free particles in terms of cartesian co-ordinates is

$$T = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_{i=1}^N \frac{1}{2} m_i (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i). \quad \dots (18)$$

Substituting for $\dot{\mathbf{r}}_i$ from equation (16), we obtain

$$\begin{aligned}T &= \sum_{i=1}^N \frac{1}{2} m_i \left[\sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right] \cdot \left[\sum_{k=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^{3N} \sum_{k=1}^{3N} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \frac{1}{2} \sum_{i=1}^N m_i \left[\sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \sum_{k=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right] \cdot \frac{\partial \mathbf{r}_i}{\partial t} \\ &\quad + \frac{1}{2} \sum_{i=1}^N m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2\end{aligned}$$

Second term on the right side consists of two sums which are identical so that

$$\begin{aligned}T &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^{3N} \sum_{k=1}^{3N} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_j \dot{q}_k + \sum_{i=1}^N \sum_{k=1}^{3N} m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \cdot \frac{\partial \mathbf{r}_i}{\partial t} + \frac{1}{2} \sum_{i=1}^N m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \\ &= T^{(2)} + T^{(1)} + T^{(0)}, \quad \dots (19)\end{aligned}$$

Thus, the general kinetic energy in terms of generalised velocities comprises three distinct terms; $T^{(2)}$ contains terms quadratic in generalised velocities and this fact is indicated by a superscript (2)

on T ; $T^{(1)}$ containing linear terms and $T^{(0)}$ is independent of generalised velocities. $T^{(1)}$ and $T^{(0)}$ will vanish when $\frac{\partial \mathbf{r}_i}{\partial t} = 0$, i.e., when our defining eq. (13) has no explicit dependence on time, i.e., when moving co-ordinate systems are not involved. In that case $T^{(2)}$ survives and contains cross terms also [like $\dot{q}_j \dot{q}_k$ ($j \neq k$)] unlike the kinetic energy (18) in cartesian co-ordinates which is purely a quadratic function of linear velocities containing only squared terms (such a form is known as *homogeneous quadratic form* or a *canonical form*). If $T^{(2)}$ is free from cross-terms which will happen when $\frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = 0$ for $j \neq k$, the generalised co-ordinate system in q 's is referred to as an *orthogonal system*.

From eq. (18) we observe that linear momentum, associated with the linear velocity $\dot{x}_i$ is $m_i \dot{x}_i$, is given by

$$p_{xi} = \frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i.$$

Then momentum associated with generalised co-ordinate q_k is similarly defined and is called the *generalised momentum* p_k associated with a co-ordinate q_k :

$$p_k = \frac{\partial T}{\partial \dot{q}_k} \quad \dots (20)$$

p_k need not always have dimensions (MLT^{-1}) of linear momentum; for, if q_k happens to be an angular co-ordinate, p_k is the corresponding angular momentum (ML^2T^{-1}). Differentiating eq. (19) w.r.t. $\dot{q}_k$ we get

$$p_k = \sum_{i=1}^N \sum_{j=1}^{3N} \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_j + \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial t} \quad \dots (21)$$

Last term will again be absent if the generalised system is stationary. It is again a linear function of generalised velocities. We shall compute generalised momenta associated with plane polar coordinates (r, θ) below:

Kinetic energy

$$T = \frac{1}{2} m (\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2), \text{ from eq. (18),}$$

and

$$\begin{aligned} K.E. = T &= \frac{1}{2} m \left(\frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial r} \right) \dot{r} \dot{r} + \frac{1}{2} m \left(\frac{\partial \mathbf{r}}{\partial \theta} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \right) \dot{\theta} \dot{\theta} + 2 \left[\frac{1}{2} m \left(\frac{\partial \mathbf{r}}{\partial r} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \right) \dot{r} \dot{\theta} \right] \\ &= \frac{1}{2} m \left[\left(\mathbf{i} \frac{\partial x}{\partial r} + \mathbf{j} \frac{\partial y}{\partial r} \right) \cdot \left(\mathbf{i} \frac{\partial x}{\partial r} + \mathbf{j} \frac{\partial y}{\partial r} \right) \right] \dot{r} \dot{r} \\ &\quad + \frac{1}{2} m \left[\left(\mathbf{i} \frac{\partial x}{\partial \theta} + \mathbf{j} \frac{\partial y}{\partial \theta} \right) \cdot \left(\mathbf{i} \frac{\partial x}{\partial \theta} + \mathbf{j} \frac{\partial y}{\partial \theta} \right) \right] \dot{\theta} \dot{\theta} \\ &\quad + 2 \left[\frac{1}{2} m \left(\mathbf{i} \frac{\partial x}{\partial r} + \mathbf{j} \frac{\partial y}{\partial r} \right) \cdot \left(\mathbf{i} \frac{\partial x}{\partial \theta} + \mathbf{j} \frac{\partial y}{\partial \theta} \right) \right] \dot{r} \dot{\theta}. \text{ from eq. (19)} \end{aligned}$$

From $x = r \cos \theta$ and $y = r \sin \theta$, we obtain

$$\frac{\partial x}{\partial r} = \cos \theta,$$

$$\frac{\partial y}{\partial r} = \sin \theta,$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta,$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta,$$

Putting them in above equation, we get for kinetic energy in generalised co-ordinates (r, θ) as:

$$\begin{aligned}
 T &= \frac{1}{2} m (\cos^2 \theta + \sin^2 \theta) \dot{r} \dot{r} + \frac{1}{2} m (r^2 \sin^2 \theta + r^2 \cos^2 \theta) \dot{\theta} \dot{\theta} \\
 &\quad + 2 \left[\frac{1}{2} m (-r \cos \theta \sin \theta + r \sin \theta \cos \theta) \dot{r} \dot{\theta} \right] \\
 &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2), \quad \dots (22)
 \end{aligned}$$

which is a familiar expression. The associated generalised momenta are given by eq. (20) as

$$\begin{aligned}
 p_r &= \frac{\partial T}{\partial \dot{r}} = m\dot{r}, & (\text{linear momentum}) \\
 p_\theta &= \frac{\partial T}{\partial \dot{\theta}} = mr^2 \dot{\theta} & (\text{angular momentum}).
 \end{aligned}$$

However, the product of any generalised momentum and the associated (also called *conjugate*) co-ordinate must always have the dimensions of angular momentum or of the dynamic variable 'action' to be defined in chapter 3.

(5) Generalised Force : The definition of generalised force associated with a generalised displacement is given as follows :

We consider the amount of work done by a force $\sum_i \mathbf{F}_i$ on the system during an arbitrary small displacement $\sum \delta \mathbf{r}_i$ of the system. We write

$$\begin{aligned}
 \delta W &= \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i \\
 &= \sum_{i=1}^N \mathbf{F}_i \cdot \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\
 &= \sum_{j=1}^{3N} Q_j \delta q_j, \quad \dots (23)
 \end{aligned}$$

where

$$Q_j = \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \quad \dots (24)$$

We note that Q_j depends on the force acting on the particles and on the co-ordinate q_j and possibly on time t . It is natural to call Q_j , the *generalised force associated with a co-ordinate q_j* ; product of Q_j with the arbitrary displacement, that a generalised co-ordinate suffers, is equal to the work done corresponding to that displacement. From eq. (23) it follows that *whatever dimension a generalised co-ordinate has, the product of the generalised force and generalised displacement (co-ordinate) must have the dimensions of work*. For that reason, *the generalised force need not always have the dimensions of force*. If we consider again a particle defined by (r, θ) generalised co-ordinates, the component of force are given by

$$\begin{aligned}
 \mathbf{F} &= F_r \hat{r} + F_\theta \hat{\theta}, & \hat{r}, \hat{\theta} \text{ are unit vectors in this case.} \\
 \mathbf{r} &= r \hat{r}.
 \end{aligned}$$

The generalised force associated with r -ordinate and θ -co-ordinate are respectively,

$$\left. \begin{aligned}
 Q_r &= \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} = (F_r \hat{r} + F_\theta \hat{\theta}) \cdot \left(\frac{\partial r}{\partial r} \hat{r} \right) = F_r, \\
 Q_\theta &= \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = (F_r \hat{r} + F_\theta \hat{\theta}) \cdot \left(r \frac{\partial \hat{r}}{\partial \theta} \right) = r F_\theta
 \end{aligned} \right\} \quad \dots (25)$$

From eq. (25), Q_r is the component of force in the r -direction and Q_θ is the *torque* acting on the particle to increase θ . That is, the generalised force associated with any co-ordinate q_k is the *force*

or the corresponding torque according as q_k measures the distance or angular displacement. Thus Q_r has dimensions of force (MLT^{-2}) where as Q_θ has dimensions of torque (ML^2T^{-2}). But the products θF_θ and rF_r always have the dimensions of work.

(6) Generalised Potential : If the forces acting on the system are derivable from scalar potential V depending on the position only, then V is the potential energy of the system and we have, for work done by the force on the system in an arbitrary displacement $\delta \mathbf{r}_i$ of the system as

$$\delta W = - \sum_{i=1}^N \left[\frac{\partial V}{\partial x_i} \delta x_i + \frac{\partial V}{\partial y_i} \delta y_i + \frac{\partial V}{\partial z_i} \delta z_i \right] \quad \dots (26)$$

$$\delta W = - \delta V = - \sum_{k=1}^{3N} \frac{\partial V}{\partial q_k} \delta q_k$$

$$= \sum_{k=1}^{3N} Q_k \delta q_k$$

$$Q_k = - \frac{\partial V}{\partial q_k} \quad \dots (27)$$

In this sense, the definition of Q_k as generalised force is a natural one. Also eq. (27), directly follows from eq. (26) : for,

$$\begin{aligned} \frac{\partial V}{\partial q_k} &= \sum_{i=1}^N \left(\frac{\partial V}{\partial x_i} \frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i} \frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i} \frac{\partial z_i}{\partial q_k} \right) \\ &= - \sum_{i=1}^N \left[\mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right] \quad \text{[From (26)]} \\ &= - Q_k \end{aligned}$$

When the system is not conservative (i.e., say depends on the generalised velocities $\dot{q}$) we define the generalised force Q_j associated with a co-ordinate q_j , not by (27) but by

$$Q_j = - \frac{\partial V}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right) \quad \dots (28)$$

where U may be called a 'velocity dependent potential' or simply 'generalised potential' since it gives rise to generalised force Q_j .

The obvious advantage that we have in formulating laws of mechanics in terms of generalised co-ordinates and the associated mechanical quantities is that the equation of motion then look simpler and can be solved independently of each other since generalised co-ordinates are all independent—constraints have no effect on them. Equations of motion are then called *Lagranges equations of motion*.

1.8. LIMITATIONS OF NEWTON'S LAWS

The direct application of Newton's laws for solving dynamical problems has many limitations :

- These laws are valid only in inertial frames which are by definition cartesian-like. For non-inertial frames such as a frame attached to earth rotating with it around its axis of rotation or the motion of any object with respect to earth is a motion in non-inertial frame of reference, a transformed version of Newton's equations of motion is obtained in which pseudoforces like coriolis forces or centrifugal forces enter into (see chapter-5).
- As these laws require complete specification of all the forces acting on a body whose motion is under consideration at all instants of time, their use poses inconvenience.

Further as they are restricted to cartesian frames, they become more inconvenient to use.

In next chapter, we opt for another type of equations of motion, called Lagrange's equation of motion which are of great importance in the advance treatment of mechanics. They can be written in any system of co-ordinates. We shall obtain them by the variational principle.

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1. *Classical Mechanics* : H. Goldstein.
2. *Mechanics* : Simon.
3. *Mechanics* : Hauser.

EXERCISES

1. What are generalised co-ordinates ? What is the advantage of using them ? Explain the statement 'generalised co-ordinates need not have dimensions of length and similarly components of generalised force do not necessarily have the dimensions of force'.
2. What are constraints ? How do they affect motion of a mechanical system ? Take specific examples to explain the forces of constraints.
3. Prove the laws of conservation of linear momentum, angular momentum and energy for a system of interacting particles.
4. Discuss the following :
 - (i) Angular momentum and kinetic energy of a system of particles can be expressed in terms of motion of centre of mass and the motion about the centre of mass.
 - (ii) It is not necessary for generalised force Q_j to have the dimensions of force but it is necessary that the product $Q_j \delta q_j$ must have the dimensions of work.
 - (iii) If equations of transformation do not involve time, the kinetic energy can be expressed as a homogeneous quadratic function of velocities.
5. Define generalised co-ordinates and obtain the expression for generalised acceleration, generalised force and generalised potential.